Unknown Input Functional Observability of Descriptor Systems with Neutral and Distributed Delay Effects[☆]

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Abstract

In this paper a general class of linear systems with time-delays is considered, which includes linear classical systems, linear systems with commensurate delays, neutral systems and singular systems with delays. After given a formal definition of functional backward observability (BO), an easily testable condition is found. The fulfillment of the obtained condition allows for the reconstruction of the trajectories of the system under consideration using the actual and past values of the system output and some of its derivatives. The methodology we follow consists in an iterative algorithm based upon the classical Silverman algorithm used for inversion of linear systems. By using basic module theory we manage to prove that the proposed algorithm is convergent. A direct application of studying functional observability is that a condition can be derived for systems with distributed delays also, we do this as a case of study. The obtained results are illustrated by two examples, one is merely academic but illustrates clearly the kind of systems which the proposed methodology works for and the other is a practical system with distributed delays.

Keywords: Descriptor systems, systems with time-delays, observability, unknown inputs, neutral delays

1. Introduction

The description of a variety of practical systems by means of descriptor systems (DS), also called singular, implicit, or differential algebraic systems, has been shown to be useful since several decades ago as it is well explained in Campbell (1980). The observability problem of DS has been studied in papers like Yip and Sincovec (1981); Cobb (1984); Hou and Müller (1999). The same problem but including unknown inputs has been addressed in Paraskevopoulos et al. (1992); Geerts (1993); Darouach et al. (1996); Koenig (2005); Darouach and Boutat-Baddas (2008); Bejarano et al. (2013). Such systems, as many others, may contain time-delay terms in the state, input, and/or system output (see, e.g. Mounier et al. (1997), Bellen et al. (1999), Zheng and Frank (2002)). Some results on the observability problem of dynamical systems with time-delays can be found in Lee and Olbrot (1981); Olbrot (1981); Malek-Zavarei (1982); Przyiuski and Sosnowski (1984); Fliess and Mounier (1998); Sename (2001); Marquez-Martinez et al. (2002); Sename (2005); Anguelova and Wennberg (2010); Zheng et al. (2011); Bejarano and Zheng (2014).

Descriptor systems with time-delays serve to describe several classes of systems, such as large scale interconnected systems, power systems, chemical processes, etc. For a more extensive revision and recent results on DS with time delays see Gu et al. (2013) and reference therein. However, despite the increasing research on problems as solvability, stability and controllability, up to the authors' knowledge, there are only few works dedicated to the study of the observability of descriptor systems with time-delays. For descriptor systems with a single time-delay in the state, a condition guaranteeing the observability of the system is given in Wei (2013). There, observability is interpreted as the reconstruction of the initial condition of the trajectories. However such a condition seems to be quite difficult to check. Two observers for particular types of linear time-delay descriptor systems with unknown inputs can be found in Perdon and Anderlucci (2006) and Khadhraoui et al. (2014). In Rabah and Sklyar (2016), the exact observability of a class of linear neutral systems is tackled. In Bejarano and Zheng (2016), the conditions are given for the observability of singular systems with commensurate time-delays. Hence, we may say that the observability problem of descriptor time-delay systems has not completely been solved.

The main motivation of this work arises from the interest of tackling the observability problem of a general class of descriptor linear time-delay systems with neutral terms, namely continuous-time systems whose dynamics is governed by equations as these ones:

$$
\begin{aligned}
J\dot{x}(t) &= \sum_{i=1}^{k_f} F_i\dot{x}(t-ih) + \sum_{i=1}^{k_a} A_i x(t-ih) + \sum_{i=1}^{k_b} B_i u(t-ih) \\
y(t) &= \sum_{i=1}^{k_c} C_i x(t-ih) + \sum_{i=1}^{k_d} D_i u(t-ih)
\end{aligned}
$$

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where the matrices J, F_i, A_i, B_i, C_i, and D_i are all constant and J is not necessarily a square matrix. It is used to defining the backward shift operator $\delta : x(t) \mapsto x(t-h)$ for rewriting the above dynamic equations as

$$\begin{aligned} J\dot{x}(t) &= F(\delta)\dot{x}(t) + A(\delta)x(t) + B(\delta)u(t) \\ y(t) &= C(\delta)x(t) + D(\delta)u(t) \end{aligned}$$

where, by definition, $F(\delta) = \sum_{i=0}^{k_f} F_i\delta^i$, $A(\delta) = \sum_{i=0}^{k_a} A_i\delta^i$, $C(\delta) = \sum_{i=0}^{k_c} C_i\delta^i$, and $D(\delta) = \sum_{i=0}^{k_d} D_i\delta^i$. The definition $E(\delta) = J - F(\delta)$ yields the following representation of the previous system equations

$$\begin{aligned} E(\delta)\dot{x}(t) &= A(\delta)x(t) + B(\delta)u(t) \\ y(t) &= C(\delta)x(t) + D(\delta)u(t) \end{aligned} \tag{1}$$

In Bejarano and Zheng (2014), the observability of the system (1) was studied considering that $E(\delta)$ is equal to an identity matrix. There sufficient conditions were given by using algebraic tools like the Smith normal form of a matrix. The aim of this paper is to extend those results to the case when the matrix $E(\delta)$ is not an invertible matrix and when only some variables are required to be reconstructed.

The following notation will be used along the paper. The limit from below of a time valued function is denoted as $f(t_-)$. $\mathbb{R}$ is the field of real numbers. $\mathbb{R}[\delta]$ is the polynomial ring over $\mathbb{R}$. I_n is the identity matrix of dimension n by n. Since hereinafter mostly matrices with terms in the polynomial ring $\mathbb{R}[\delta]$ will be needed, instead of using the symbol (δ) behind a matrix to indicate that its elements are within $\mathbb{R}[\delta]$, we prefer to use a more compact notation. As for, we express the polynomial ring as $\mathfrak{R} = \mathbb{R}[\delta]$. Thus, $\mathfrak{R}^{r\times s}$ means the set of all matrices whose dimension is r by s and whose entries are within $\mathfrak{R}$. A square matrix T whose terms belong to $\mathfrak{R}$ is called unimodular (or invertible) if its determinant is a nonzero constant. A matrix $M \in \mathfrak{R}^{r\times s}$ is called left unimodular (invertible) if there exists a matrix $M^+ \in \mathfrak{R}^{s\times r}$ such that $M^+M = I_r$. For a matrix F (with terms in $\mathfrak{R}$), $\operatorname{rank} F$ denotes the rank of F over $\mathfrak{R}$. The degree of a polynomial $p(\delta)$ is denoted by $\deg p(\delta)$. For a matrix F with elements in $\mathfrak{R}$, $\deg F$ denotes the greatest degree of all entries of F. By $\operatorname{Inv_s} F$ we denote the set of invariant factors of the matrix F.

2. Formulation of the problem

We consider the class of systems that can be represented by the following equations

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{2a}$$
$$y(t) = Cx(t) + Du(t) \tag{2b}$$
$$z(t) = \Psi x(t) \tag{2c}$$

where, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$, and $z(t) \in \mathbb{R}^q$. The function $u(t)$ is assumed to be **unknown**, but piecewise continuous. The vector $z(t)$ is attempted to be reconstructed. The dimension of the matrices is as follows, $E \in \mathfrak{R}^{\bar{n}\times n}$, $A \in \mathfrak{R}^{\bar{n}\times n}$, $B \in \mathfrak{R}^{\bar{n}\times m}$, $C \in \mathfrak{R}^{p\times n}$, $D \in \mathfrak{R}^{p\times m}$, $\Psi \in \mathfrak{R}^{q\times n}$ (with $\bar{n} \leq n$). Notice that E does not have to be a square matrix. According to the notation defined previously, where $\mathfrak{R} = \mathbb{R}[\delta]$, we use δ as the shift backward operator, i.e., $\delta : x(t) \mapsto x(t-h)$, where h is a non negative real number. The initial condition $\varphi(t)$ is a piecewise continuous function $\varphi : [-kh, 0] \to \mathbb{R}^n$, where k is the greatest degree of all polynomial terms of the matrices involved in system (2), hence $x(t) = \varphi(t)$ on $[-kh, 0]$. For $x(t;\varphi,u)$ we mean the solution of (2a) (assuming it exists) for the initial condition $\varphi = \varphi(t)$ and the input $u = u(t)$; $y(t;\varphi,u)$ and $z(t;\varphi,u)$ are defined analogously by (2b), (2c).

It is assumed that system (2a) admits at least one solution. However, it is worth noticing that the solvability and the observability problems are not related to each other, that is, the system could have more than one solution but still may be observable. That is why, for the observability analysis to be carried out, the equation (2a) is allowed for having more than one solution. We assume also that any solution of (2a) is piecewise differentiable.

The following definition is taken as the starting point for the observability analysis that will be done further.

Definition 1. The vector $z(t)$ in (2) is called backward unknown input observable (**BUIO**) on $[t_1, t_2]$ if for every $\tau \in [t_1, t_2]$ there exist $\bar{t}_1$ and $\bar{t}_2$, with $\bar{t}_1 < \bar{t}_2 \leq \tau$, such that, for any input $u(t)$ and any initial condition $\varphi(t)$, the identity $y(t;\varphi,u) = 0$ for all $t \in [\bar{t}_1, \bar{t}_2]$ is true only if $z(\tau_-;\varphi,u) = 0$.

The above definition of backward observability is related to the final observability given in Lee and Olbrot (1981), and it means that the reconstruction of $x(t)$ depends only on previous and actual values of $y(t)$ and some of its derivatives.

3. Like Silverman-Molinari algorithm

The BUIO will be checked by means of the matrix N_{k^*} which will be defined further. Firstly, let us select a unimodular matrix $S_0 \in \mathfrak{R}^{\bar{n}\times\bar{n}}$ so that we obtain the identity,

$$S_0 \begin{bmatrix} -I & E \end{bmatrix} = \begin{bmatrix} J_0 & R_0 \\ H_0 & 0 \end{bmatrix} \text{ such that } R_0 \in \mathfrak{R}^{\beta_0\times n} \tag{3}$$

where $\beta_0 = \operatorname{rank}(E)$.

Now, let us consider the following matrices, $G_0 = C$, $F_0 = D$. For the k-th step ($k \geq 1$) the matrices Δ_k, N_k and H_k are generated by using the following general procedure. Let us select a unimodular matrix T_k so that

$$T_k \begin{bmatrix} H_{k-1}A & H_{k-1}B \\ G_{k-1} & F_{k-1} \end{bmatrix} = \begin{bmatrix} G_k & F_k \\ \Delta_k & 0 \end{bmatrix} \text{ such that } F_k \in \mathfrak{R}^{\alpha_k\times m} \tag{4}$$

where $\alpha_k = \operatorname{rank} \begin{bmatrix} H_{k-1}B \\ F_{k-1} \end{bmatrix}$. With the matrix Δ_k defined explicitly by equation (4), a new matrix, denoted as N_k, is formed with the concatenation of the matrices $\Delta_1, \Delta_2, \ldots, \Delta_k$, that is,

$$N_k = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_k \end{bmatrix}. \tag{5}$$

The construction of the matrix H_k is done by means of the following equation

$$S_k \begin{bmatrix} -I & E \\ 0 & N_k \end{bmatrix} = \begin{bmatrix} J_k & R_k \\ H_k & 0 \end{bmatrix} \text{ such that } R_k \in \Re^{\beta_k \times n} \tag{6}$$

where $\beta_k = \operatorname{rank} \begin{bmatrix} E \\ N_k \end{bmatrix}$. The matrix S_k must be chosen to be unimodular and so that (6) is satisfied.

Proposition 1. *The matrix H_0 satisfies the equation $H_0 E = 0$ and for $k \geq 1$, there exists a matrix Γ_k that satisfies the equation*

$$H_k E = \Gamma_k N_k \tag{7}$$

Furthermore, if $HE = \Gamma N_k$, for some matrices H and Γ, then $\operatorname{rank} HE \leq \operatorname{rank} H_k E$.

Proof. Let us give a partition of S_k as $S_k = \begin{bmatrix} S_{k,1} & S_{k,2} \\ S_{k,3} & S_{k,4} \end{bmatrix}$. Then, by (6), we obtain

$$\begin{bmatrix} -S_{k,1} & S_{k,1}E + S_{k,2}N_k \\ -S_{k,3} & S_{k,3}E + S_{k,4}N_k \end{bmatrix} = \begin{bmatrix} J_k & R_k \\ H_k & 0 \end{bmatrix}$$

Thereby, $H_k = -S_{k,3}$ and so $-H_k E + S_{k,4} N_k = 0$. Comparing the last equation with (7), we see that $S_{k,4}$ acts as the matrix Γ_k we were seeking.

Now, since the rank of the matrix $\begin{bmatrix} H_k & \Gamma_k \end{bmatrix}$ is equal to the dimension of the left null space of the matrix $\begin{bmatrix} E \\ N_k \end{bmatrix}$, then, among all matrices Γ and H satisfying the identity $HE = \Gamma N_k$, matrices H_k and Γ_k are the ones for which the rank of $H_k E$ is the maximum. □

Lemma 1. *There exists a unimodular matrix W_k such that the matrix N_k (for $k \geq 1$), generated by (5), satisfies the identity*

$$W_k \begin{bmatrix} L_{k-1}A & L_{k-1}B \\ C & D \end{bmatrix} = \begin{bmatrix} G_k & F_k \\ N_k & 0 \end{bmatrix} \tag{8}$$

where $L_{k-1} = \begin{bmatrix} H_{k-1} \\ \vdots \\ H_1 \\ H_0 \end{bmatrix}$.

Proof. Since $N_1 = \Delta_1$ and $L_0 = H_0$, then, in view of (4), we can take $W_1 = T_1$. Now, let us suppose that (8) is valid for $k = i \geq 1$. By using first (8) and then (4), we obtain the following identity

$$\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} T_{i+1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_i \end{bmatrix}}_{W_{i+1}} \times$$

$$\times \begin{bmatrix} H_i A & H_i B \\ L_{i-1}A & L_{i-1}B \\ C & D \end{bmatrix} = \begin{bmatrix} G_{i+1} & F_{i+1} \\ N_i & 0 \\ \Delta_{i+1} & 0 \end{bmatrix}$$

Taking W_{i+1} as above, the equation in (8) is satisfied for $k = i+1$. Thus, the lemma has been proved by induction. □

Theorem 1. *There exists a positive integer k^* with the following properties:*

1. k^* and the set $\operatorname{Inv_s}(N_{k^*})$ are independent of the choices of the matrices $\{S_i\}$ and $\{T_{i+1}\}$ for $i = 0, 1, 2, \ldots, k^*$.
2. $\operatorname{Inv_s}(N_{k^*+i}) = \operatorname{Inv_s}(N_{k^*})$, for all $i \geq 1$.
3. k^* is the least positive integer for which the identity $\operatorname{Inv_s} N_{k^*+1} = \operatorname{Inv_s} N_{k^*}$ is satisfied.

Proof. (Clause 1) We will prove it by induction. Let us select a set of matrices $\{S_i\}$ and $\{T_{i+1}\}$, $i = 0, 1, 2, \ldots, k$ ($k > 1$) following the algorithm (4)-(6) iteratively. Now, we follow the same algorithm, but with a different choice of matrices, let us say $\{\bar{S}_i\}$ and $\{\bar{T}_{i+1}\}$, $i = 0, 1, 2, \ldots, k$ ($k > 1$). Thus, the over-lined matrices allude to the matrices obtained by (4)-(6) with the use of $\{\bar{S}_i\}$, and $\{\bar{T}_{i+1}\}$, $i = 0, 1, 2, \ldots, k$ ($k > 1$). Let S_0 and $\bar{S}_0$ be two matrices used in (3) to generate H_0 and $\bar{H}_0$, respectively. Hence, the following equation is readily obtained,

$$\begin{bmatrix} J_0 & R_0 \\ H_0 & 0 \end{bmatrix} = S_0 \left(\bar{S}_0^{-1} \bar{S}_0\right) \begin{bmatrix} -I & E \end{bmatrix} = \underbrace{S_0 \bar{S}_0^{-1}}_{S_0^*} \begin{bmatrix} \bar{J}_0 & \bar{R}_0 \\ \bar{H}_0 & 0 \end{bmatrix} \tag{9}$$

Let us partition S_0^* as $S_0^* = \begin{bmatrix} \Xi & \Upsilon \\ \Omega & \Theta_0 \end{bmatrix}$. Equation (9) yields the identities $H_0 = \Omega \bar{J}_0 + \Theta_0 \bar{H}_0$, $0 = \Omega \bar{R}_0$. Regarding that $\bar{R}_0$ has full row rank, by the second of the later equations Ω must be equal to zero. This has two implications, Θ_0 must be unimodular since S_0^* is unimodular too and $H_0 = \Theta_0 \bar{H}_0$.

Moreover, let T_1 and $\bar{T}_1$ be two matrices used in (4) to generate Δ_1 and $\bar{\Delta}_1$, respectively. Then

$$\begin{bmatrix} G_1 & F_1 \\ \Delta_1 & 0 \end{bmatrix} = T_1 \begin{bmatrix} \Theta_0 & 0 \\ 0 & I \end{bmatrix} \bar{T}_1^{-1} \begin{bmatrix} \bar{G}_1 & \bar{F}_1 \\ \bar{\Delta}_1 & 0 \end{bmatrix} \tag{10}$$

Since both F_1 and $\bar{F}_1$ have the same rank, which is equal to their number of rows, then there exists a unimodular matrix Ω_1 such that $\Delta_1 = \Omega_1 \bar{\Delta}_1$.

Now, let us assume that for $i \geq 1$ there exists a unimodular matrix Ω_j such that $N_j = \Omega_j \bar{N}_j$, for all positive integer $j \leq i$. Then, in view of (6), the following equation can be easily verified,

$$\begin{bmatrix} J_j & R_j \\ H_j & 0 \end{bmatrix} = S_j \begin{bmatrix} I & 0 \\ 0 & \Omega_j \end{bmatrix} \bar{S}_j^{-1} \begin{bmatrix} \bar{J}_j & \bar{R}_j \\ \bar{H}_j & 0 \end{bmatrix}$$

The previous equation yields the existence of a unimodular matrix Θ_j such that $H_j = \Theta_j \bar{H}_j$, for all $j \leq i$. Hence, there exists a unimodular matrix, formed with the matrices $\{\Theta_j\}$ such that $L_i = \Phi_i \bar{L}_i$. Thus, according to Lemma 1, the following equation is obtained,

$$\begin{bmatrix} G_{i+1} & F_{i+1} \\ N_{i+1} & 0 \end{bmatrix} = W_{i+1} \begin{bmatrix} \Phi_i & 0 \\ 0 & I \end{bmatrix} \bar{W}_{i+1}^{-1} \begin{bmatrix} \bar{G}_{i+1} & \bar{F}_{i+1} \\ \bar{N}_{i+1} & 0 \end{bmatrix}$$

Thus, again, $N_{i+1} = \Omega_{i+1} \bar{N}_{i+1}$ for a unimodular matrix Ω_{i+1}.

To finish, in view of the last equation, we have proved by induction that the set of invariant factors of the matrix N_k ($k \geq 1$) is independent of the choices of matrices $\{S_j\}$ for $j = 0, 1, 2, \ldots k$ and $\{T_j\}$ for $j = 1, 2, \ldots k$.

(Clause 2) Let us define $\mathscr{N}_k$ as the $\mathfrak{R}$-module generated by the rows of N_k. By the way the matrices $\{N_k\}$ were generated, the respective modules $\{\mathscr{N}_k\}$ satisfy the ascending chain

$$\mathscr{N}_0 \subset \mathscr{N}_1 \subset \cdots \subset \mathscr{N}_k \subset \mathscr{N}_{k+1} \subset \cdots$$

By its definition every $\mathscr{N}_k$ is a submodule of the module $\mathfrak{R}^{1\times n}$, which is a Noetherian module (see e.g. Proposition 6.5 in Atiyah and Macdonald (1994)). Thereby, the above chain is stationary, that is, there exists a least positive integer, let say k^*, such that $\mathscr{N}_{k^*+i} = \mathscr{N}_{k^*}$, for any $i \geq 0$, which in turn implies that N_{k^*} and N_{k^*+i} have both the same invariant factors.

(Clause 3) By the proof of clause 2, there exists k such that $\mathrm{Inv_s}(N_{k+1}) = \mathrm{Inv_s}(N_k)$. Now, let us assume that $\mathrm{Inv_s}\left(N_{k+j+1}\right) = \mathrm{Inv_s}\left(N_{k+j}\right)$ for a non negative integer j. The last equation is equivalent to $\mathscr{N}_{k+j+1} = \mathscr{N}_{k+j}$, which in turn implies that the rows of Δ_{k+j+1} are linearly dependent of the rows of N_{k+j}, that is, $\Delta_{k+j+1} = X_{k+j+1}N_{k+j}$ for some matrix X_{k+j+1}. In view of the previous identity, we obtain the equation

$$\underbrace{\begin{bmatrix} S_{k+j} & 0 \\ \begin{bmatrix} 0 & -X_{k+j} \end{bmatrix} & I \end{bmatrix}}_{S_{k+j+1}} \begin{bmatrix} -I & E \\ 0 & N_{k+j} \\ 0 & \Delta_{k+j+1} \end{bmatrix} = \begin{bmatrix} J_{k+j} & R_{k+j} \\ H_{k+j} & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, with S_{k+j+1} defined as above, a matrix H_{k+j+1} obtained by (6) takes the form $H_{k+j+1} = \begin{bmatrix} H_{k+j} \\ 0 \end{bmatrix}$, which in turn implies that there exists a matrix Y_{k+j+1} such that $H_{k+j+1} = Y_{k+j+1}L_{k+j}$ (L_{k+j} was defined right after (8)). Thus, according to (8), we obtain the equation

$$\underbrace{\begin{bmatrix} 0 & W_{k+j+1} \\ I & \begin{bmatrix} - & Y_{k+j+1} & 0 \end{bmatrix} \end{bmatrix}}_{T'} \underbrace{\begin{bmatrix} H_{k+j+1}A & H_{k+j+1}B \\ L_{k+j}A & L_{k+j}B \\ C & D \end{bmatrix}}_{\Theta} = \begin{bmatrix} G_{k+j+1} & F_{k+j+1} \\ N_{k+j+1} & 0 \\ 0 & 0 \end{bmatrix}$$

On the other hand, by Lemma 1, $W_{k+j+2}\Theta = \begin{bmatrix} G_{k+j+2} & F_{k+j+2} \\ N_{k+j+2} & 0 \end{bmatrix}$. Since F_{k+j+1} and F_{k+j+2} have both the same rank (equal to their number of rows) and T' and W_{k+j+2} are both unimodular, then it is easy to check that there exists a unimodular matrix Λ such that

$$N_{k+j+2} = \Lambda \begin{bmatrix} N_{k+j+1} \\ 0 \end{bmatrix}$$

which means that the rows of N_{k+j+1} and the rows N_{k+j+2} generate each one the same module, consequently, both matrices have the same invariant factors. Thus, we have proved by induction that if $\mathrm{Inv_s}(N_{k+1}) = \mathrm{Inv_s}(N_k)$ then $\mathrm{Inv_s}(N_{k+i}) = \mathrm{Inv_s}(N_k)$ for all $i \geq 1$. Therefore, since k^* is the least positive integer for which $\mathrm{Inv_s}(N_{k+i}) = \mathrm{Inv_s}(N_k)$ for all $i \geq 1$, then k^* is also the least positive integer that satisfies $\mathrm{Inv_s}(N_{k^*+1}) = \mathrm{Inv_s}(N_{k^*})$. $\square$

4. Observability sufficient conditions

In this section we give sufficient conditions guaranteeing the BUI observability of the system in an interval $[t^*, \infty)$.

Lemma 2. *The identity $y(t) = 0$ for all $t \in [0, \tau]$ implies $N_{k^*}x(t) = 0$ for all $t \in [t^*, \tau)$, for any $\tau > t^* \triangleq \alpha_{k^*}h$ ($\alpha_{k^*} = \max_{1\leq i\leq k^*} \{\deg H_{i-1}, \deg T_i\}$).*

Proof. Let us take for granted that $y(t) = 0$ for all $t \in [0, \tau]$. The following equation is obtained readily by (4), and the fact that $H_0E = 0$,

$$T_1 \begin{bmatrix} H_0E\dot{x}(t) \\ y(t) \end{bmatrix} = T_1 \begin{bmatrix} H_0Ax(t) + H_0Bu(t) \\ Cx(t) + Du(t) \end{bmatrix} = \begin{bmatrix} G_1x(t) + F_1u(t) \\ \Delta_1x(t) \end{bmatrix} = 0$$

for all $t \in [\alpha_1 h, \tau]$. Thus, by using Proposition 1, we can derive the following equation,

$$\xi_1(t) \triangleq \frac{d}{dt}\Gamma_1\Delta_1x(t) = H_1(Ax(t) + Bu(t)) = 0$$

for all $t \in [\beta_1 h, \tau)$ ($\beta_1 = \max\{\alpha_1, \deg H_1\}$). After substituting the terms of the above equation into (4) for $k = 2$, we get

$$T_2 \begin{bmatrix} \xi_1(t) \\ G_1x(t) + F_1u(t) \end{bmatrix} = \begin{bmatrix} G_2x(t) + F_2u(t) \\ \Delta_2x(t) \end{bmatrix} = 0$$

for all $t \in [\alpha_2 h, \tau)$ ($\alpha_2 = \max_{i=1,2}\{\deg H_{i-1}, \deg T_i\}$). Following the same procedure iteratively, we obtain the following identities, for all $t \in [\alpha_k h, \tau)$ ($\alpha_k = \max_{1\leq i\leq k}\{\deg H_{i-1}, \deg T_i\}$),

$$\xi_{k-1}(t) \triangleq \frac{d}{dt}\Gamma_{k-1}N_{k-1}x(t) = H_{k-1}(Ax(t) + Bu(t)) = 0 \quad (11)$$

and the equation

$$T_k \begin{bmatrix} \xi_{k-1}(t) \\ G_{k-1}x(t) + F_{k-1}u(t) \end{bmatrix} = \begin{bmatrix} G_kx(t) + F_ku(t) \\ \Delta_kx(t) \end{bmatrix} = 0 \quad (12)$$

for all $k \geq 2$. Therefore, the end of the proof follows from the equations (5) and (12). $\square$

Theorem 2. *The vector $z(t)$ is BUIO on $[t_1^*, \infty)$ ($t_1^* = t^* + \deg\Psi$) provided that $\mathrm{Inv_s}\begin{pmatrix} N_{k^*} \\ \Psi \end{pmatrix} = \mathrm{Inv_s}\, N_{k^*}$.*

Proof. The identity $\mathrm{Inv_s}\begin{pmatrix} N_{k^*} \\ \Psi \end{pmatrix} = \mathrm{Inv_s}\, N_{k^*}$ implies that there exists a matrix Υ such that $\Upsilon N_{k^*} = \Psi$ (see, e.g. Gohberg et al. (2009)). Thus, $N_{k^*}x(t) = 0$ for all $t \in [t^*, \tau)$, implies that $\Psi x(t) = 0$ for all $t \in [t^* + \deg\Psi, \tau)$. Hence, we obtain the identity $\Psi x(\tau) = 0$, which, in view of Lemma 2, proves the theorem. $\square$

Corollary 1. *The vector $x(t)$ is BUIO on $[t_1^*, \infty)$ if N_{k^*} is left invertible (i.e. if N_{k^*} has n real non zero invariant factors).*

Knowing that whether the condition given in Theorem 2 is satisfied or not depends upon building a matrix N_{k^*}, which can be done by using a computer program in a software able to carry out symbolic calculations. For that purpose, a pseudocode with the procedure that should be followed is given in Algorithm 1.

Algorithm 1 Checking the observability condition

procedure SILVERMAN-MOLINARI(E,A,B,C,D) ▷ Finding N_{k^*} and its invariant factors.
  $\beta \leftarrow \operatorname{rank}(E)$
  $S \leftarrow$ a unimodular matrix so that $SE = \begin{bmatrix} R \\ 0 \end{bmatrix}$ and $\operatorname{rank}(R) = \beta$
  $S_2 \leftarrow$ a matriz obtained by splitting $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ so that $S_1 \in \mathfrak{R}^{\beta \times \bar{n}}$
  $H \leftarrow -S_2$
  $\alpha \leftarrow rank \begin{bmatrix} HB \\ D \end{bmatrix}$
  $T \leftarrow$ a matrix so that $T \begin{bmatrix} HB \\ D \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$ and $\operatorname{rank} F = \alpha$
  $\begin{bmatrix} G \\ \Delta \end{bmatrix} \leftarrow T \begin{bmatrix} HA \\ C \end{bmatrix}$
  $inv1 \leftarrow 0$
  $inv2 \leftarrow$ a vector with the invariant factors of Δ
  while $inv1 \neq inv2$ **do** ▷ The algorithm stops when the invariant factors do not change respect to the previous iteration.
    $inv1 \leftarrow inv2$
    $\alpha \leftarrow rank \begin{bmatrix} HB \\ F \end{bmatrix}$
    $T \leftarrow$ a matrix so that $T \begin{bmatrix} HB \\ F \end{bmatrix} = \begin{bmatrix} F_k \\ 0 \end{bmatrix}$ and $\operatorname{rank} F_k = \alpha$
    $\begin{bmatrix} G_k \\ \Delta \end{bmatrix} \leftarrow T \begin{bmatrix} HA \\ G \end{bmatrix}$
    $N_k \leftarrow \begin{bmatrix} N \\ \Delta \end{bmatrix}$
    $\beta \leftarrow \operatorname{rank}\left(\begin{bmatrix} E \\ N_k \end{bmatrix}\right)$
    $S \leftarrow$ a matrix so that $S \begin{bmatrix} E \\ N_k \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}$ and $\operatorname{rank} R = \beta$ ▷ S must be unimodular.
    $S_3 \leftarrow$ a matrix obtained by spliting $S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$ so that $S_1 \in \mathfrak{R}^{\rho \times \bar{n}}$
    $H \leftarrow -S_3$
    $F \leftarrow F_k$
    $G \leftarrow G_k$
    $N \leftarrow N_k$
    $inv2 \leftarrow$ a vector with the invariant factors of N.
  end while
  return N and $inv2$ ▷ The matrix N_{k^*} and a vector with the invariant factors of it.
end procedure
$inv3 \leftarrow$ a vector with the invariant factors of $\begin{bmatrix} N \\ \Psi \end{bmatrix}$
if $inv2 = inv3$ **then**
  $z(t)$ **is BUIO**
end if

5. Case of study: Descriptor neutral systems with distributed delays

In this section we will apply the results on functional observability achieved in the previous section to infer BUI observability conditions for descriptor systems with distributed and commensurate delays (see, e.g. Adimy et al. (2008)). Let us consider the sort of systems governed by the following equations,

$$\bar{E}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) + \sum_{i=1}^{k_f} \int_{t-ih}^{t} \bar{F}_i \bar{x}(\xi)\, d\xi + \sum_{i=1}^{k_g} \int_{t-ih}^{t} \bar{G}_i u(\xi)\, d\xi \tag{13a}$$

$$y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t) + \sum_{i=1}^{k_h} \int_{t-ih}^{t} \bar{H}_i \bar{x}(\xi)\, d\xi + \sum_{i=1}^{k_j} \int_{t-ih}^{t} \bar{J}_i u(\xi)\, d\xi \tag{13b}$$

where the above matrices $\bar{E}$, $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ have terms in the polynomial ring $\mathfrak{R}$ as in (2a)-(2b). The matrices $\bar{F}_i$, $\bar{G}_i$, $\bar{H}_i$, and $\bar{J}_i$ have all their terms in the real field $\mathbb{R}$. Let us notice that with the notation used so far, the following equality is fulfilled,

$$\int_{t-ih}^{t} = \sum_{j=1}^{i} \int_{t-jh}^{t-(j-1)h} = \sum_{j=0}^{i-1} \delta^j \int_{t-h}^{t} \tag{14}$$

Let us define the matrices $\bar{F} = \sum_{i=1}^{k_f} \bar{F}_i \left(\sum_{j=0}^{i-1} \delta^j \right)$, $\bar{G} = \sum_{i=1}^{k_g} \bar{G}_i \sum_{j=0}^{i-1} \delta^j$, $\bar{H} = \sum_{i=1}^{k_h} \bar{H}_i \sum_{j=0}^{i-1} \delta^j$, and $\bar{J} = \sum_{i=1}^{k_j} \bar{J}_i \sum_{j=0}^{i-1} \delta^j$. Hence, taking into account (14), the system (13a)-(13b) can be rewritten

$$\bar{E}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{F} \int_{t-h}^{t} \bar{x}(\xi)\, d\xi + \bar{G} \int_{t-h}^{t} u(\xi)\, d\xi$$
$$y(t) = \bar{C}\bar{x}(t) + \bar{D}u(t) + \bar{H} \int_{t-h}^{t} \bar{x}(\xi)\, d\xi + \bar{J} \int_{t-h}^{t} u(\xi)\, d\xi$$

as

Let us define vectors $v(t) = \int_{t-h}^{t} \bar{x}(\xi)\, d\xi$ and $w(t) = \int_{t-h}^{t} u(\xi)\, d\xi$. Thereby, taking into account that $\dot{v}(t) = (1-\delta)\bar{x}(t)$ and $\dot{w}(t) = (1-\delta)u(t)$, we get the following extended system:

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{F}v(t) + \bar{G}w(t) + \bar{B}u(t) \\ \dot{v}(t) &= (1-\delta)\bar{x}(t) \\ \dot{w}(t) &= (1-\delta)u(t) \\ y(t) &= \bar{C}\bar{x}(t) + \bar{H}v(t) + \bar{J}w(t) + \bar{D}u(t) \end{aligned}$$

The previous equations can be put into the form (2), by defining the extended vector $x=\left(\bar{x}^T,v^T,w^T\right)^T$, i.e.,

$$E\dot{x}(t)=Ax(t)+Bu(t) \tag{15a}$$
$$y=Cx(t)+Du(t) \tag{15b}$$
$$\bar{x}=\Psi x \tag{15c}$$

where

$$E=\begin{pmatrix}\bar{E} & 0 & 0\\ 0 & I_n & 0\\ 0 & 0 & I_m\end{pmatrix}, A=\begin{pmatrix}\bar{A} & \bar{F} & \bar{G}\\ (1-\delta)I_n & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$
$$B=\begin{pmatrix}\bar{B}\\ 0\\ (1-\delta)I_m\end{pmatrix}$$
$$C=\begin{pmatrix}\bar{C} & \bar{H} & \bar{J}\end{pmatrix}, D=\bar{D}, \Psi=\begin{pmatrix}I_n & 0 & 0\end{pmatrix}$$

Hence, conditions allowing for the reconstruction of the vector $\bar{x}(t)$ can be derived by Theorem 2 straightforwardly

Theorem 3. *The vector $\bar{x}(t)$ of the system (13) is BUIO on $[t^*,\infty)$ if* $\mathrm{Inv_s}\begin{pmatrix}N_{k^*}\\ \begin{bmatrix}I_n & 0 & 0\end{bmatrix}\end{pmatrix}=\mathrm{Inv_s}\,N_{k^*}$.

Proof. After rewriting (13) into the form (15), the proof follows directly by applying Theorem 2 . □

6. Examples

Example 1. Let us consider the following academic example,

$$\dot{\bar{x}}_1(t)=\bar{x}_2(t)-\bar{x}_1(t)+\bar{x}_4(t)+\int_{t-h}^{t}\bar{x}_5(\xi)\,d\xi+$$
$$+\int_{t-2h}^{t}u_1(\xi)\,d\xi$$
$$\dot{\bar{x}}_2(t)=\bar{x}_2(t)-\bar{x}_1(t)+\bar{x}_4(t)+\delta\bar{x}_5(t)+\int_{t-2h}^{t}u_1(\xi)\,d\xi$$
$$\dot{\bar{x}}_4(t)=\int_{t-h}^{t}\bar{x}_5(\xi)+\int_{t-2h}^{t}u_1(\xi)\,d\xi$$
$$\dot{\bar{x}}_5(t)=(1-\delta)^2(\bar{x}_1(t)-\bar{x}_4(t))-\bar{x}_5(t)$$

$$\dot{\bar{x}}_6(t)=\int_{t-h}^{t}\bar{x}_5(\xi)-\bar{x}_4(t)+\int_{t-2h}^{t}u_1(\xi)\,d\xi$$
$$0=\bar{x}_3(t)-\bar{x}_4(t)+\bar{x}_6(t)-u_2(t)$$

with the following system outputs

$$y_1(t)=\bar{x}_1(t)-\bar{x}_4(t)$$
$$y_2(t)=\bar{x}_4(t)-\bar{x}_6(t)$$
$$y_3(t)=\bar{x}_3(t)-\bar{x}_4(t)+\bar{x}_6(t)+u_2(t)$$

Following the procedure described in (5) to calculate N_{k^*} and the third clause of Theorem 1 (see also Algorithm 1), we obtain that $k^*=4$, and N_{k^*} takes the form $N_{k^*}=\begin{bmatrix}\bar{N} & \tilde{N}\end{bmatrix}$, with

$$\bar{N}=\begin{bmatrix}0 & 0 & 1 & -1 & 0 & 1\\ 1 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1 & 0 & 1\\ -1 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\\ 0 & 0 & 0 & 0 & \delta & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\\ \delta(1-\delta)^2 & 0 & 0 & -\delta(1-\delta)^2 & -\delta & \delta-1\end{bmatrix}$$

$$\tilde{N}=\left[\begin{array}{c|cccc}0_{7\times4} & & 0_{7\times4} & & \\ \hline & 0 & 0 & 1+\delta & 0\\ 0_{4\times4} & -1 & 0 & \delta-1 & 0\\ & 0 & 0 & 0 & 0\\ & 0 & 0 & 0 & 0\end{array}\right]$$

Condition of Theorem 3 is satisfied, that is

$$\mathrm{Inv_s}\begin{pmatrix}N_{k^*}\\ \begin{bmatrix}I_n & 0 & 0\end{bmatrix}\end{pmatrix}=\mathrm{Inv_s}(N_{k^*})=\{1,1,1,1,1,1,1,1+\delta\}$$

Therefore, according to the that theorem, the vector $\bar{x}(t)$ is BUIO. In fact, the states can be explicitly expressed in terms of the system output and some of its derivatives by means of the following formula,

$$\bar{x}_1(t)=y_1(t)+\dot{y}_2(t)$$
$$\bar{x}_2(t)=y_1(t)+\dot{y}_1(t)$$
$$\bar{x}_3(t)=\frac{1}{2}[y_3(t)-y_2(t)]$$
$$\bar{x}_4(t)=\dot{y}_2(t)$$
$$\bar{x}_5(t)=\delta(1-\delta)^2y_1(t)+\dddot{y}_2(t)-\dddot{y}_1(t)$$
$$\bar{x}_6(t)=\dot{y}_2(t)-y_2(t)$$

Example 2. Let us consider the following linearized equations of a liquid monopropellant rocket motor with a pressure feeding system (Zheng and Frank (2002))

$$\dot{x}(t)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & -5.0\\ -0.5556 & 0 & -0.556 & 0.5556\\ 0 & 1 & -1 & 0\end{bmatrix}x(t)$$
$$+\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\int_{t-3h}^{t}x(\tau)\,d\tau+\begin{bmatrix}0\\ 5.0\\ 0\\ 0\end{bmatrix}u(t)$$
$$y(t)=\begin{bmatrix}x_1(t)\\ x_3(t-h)\end{bmatrix}$$

where $h = \frac{1}{3}$, $x(t) = \left[\begin{array}{cccc} \phi(t) & \mu_1(t) & \mu(t) & \psi(t) \end{array} \right]$, $\phi(t)$ is a variable related with the instantaneous pressure in the combustion chamber, $\mu_1(t)$ is a variable related with the instantaneous mass flow upstream of the capacitance, $\mu(t)$ is a variable related with the instantaneous mass rate of the injected propellant, and $\psi(t)$ is a variable related with the instantaneous pressure in the feeding line where the capacitance representing the elasticity is located. We refer the reader to Zheng and Frank (2002) for further explanation of the system. Following the procedure given in Section 5 to rewrite the equations into the form (15) and then using Algorithm 1, we obtain that $\mathrm{Inv_s}\, N_{k*} = \mathrm{Inv_s} \left(\begin{array}{c} N_{k^*} \\ \left[\begin{array}{cc} I_4 & 0 \end{array} \right] \end{array} \right) = \{1,1,1,1\}$. Therefore, the system is BUIO. In fact, as we see below, each variable of $x(t)$ can be expressed as a function of the system output and its derivatives,

$$\phi(t) = y_1(t)$$

$$\mu_1(t) = 1.799 y_1^{(4)}(t) + 1.799\left(1+\delta^2\right)\ddot{y}_1(t) + 1.799\delta^2\ddot{y}_2(t) - \left(2-\delta^3\right)\dot{y}_1(t) + \delta^2\dot{y}_2(t) - \left(1-\delta^3\right)y_1(t) - \delta^3 y_2(t)$$

$$\mu(t) = \ddot{y}_1(t) + \left(1-\delta^3\right)y_1(t) + \delta^2 y_2(t)$$

$$\psi(t) = 1.799 y_1^{(3)}(t) + \left(2.799 + 1.799\delta^2\right)\dot{y}_1(t) + 1.799\delta^2\dot{y}_2(t) + \left(2-\delta^3\right)y_1(t) + \delta^2 y_2(t)$$

Conclusions

We have proposed new conditions guaranteeing the functional BUI observability of a general class of linear systems with time-delays. The obtained conditions can be verified by checking the invariant factors of the matrix N_{k^*}. We have shown that N_{k^*} is obtained by a finite number of steps of the proposed algorithm. Such a matrix can be calculated easily by using any software commonly used to do computations with polynomial matrices. We have shown the usefulness of introducing the time-shift delay operator for the distributed terms, in such a way we can define an extended vector without distributed terms, which allows for solving the observability problem of the vector by applying to an extended vector the results obtained in functional BUI observability. It is worth noticing that the obtained conditions are also necessary when all matrices in (2) are real (see Bejarano et al. (2013)).

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